A GEOMETRICAL APPLICATION OF BINARY SYZYGIES*

BY

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Introductory. The Jonquières curve.

1. The present paper is devoted to a geometrical application of certain kinds of syzygies connected with binary forms of odd order. If we represent the forms with which we deal by sets of points on a rational curve, the application is to the problem of determining other curves of the same type whose intersections with the given curve are either original sets themselves, or new sets having simple invariant relations to the old.

The type of rational curve which we shall use is that having but one multiple point, which then, for a curve of order m, must be an $(m-1)$-fold point, and equivalent to $\frac{1}{2}(m-1)(m-2)$ double points. Such a curve we shall designate for brevity as a JONQUIÈRES curve, and the symbol $J^{(m)}$ will be used in referring to it.

We proceed to consider the intersections — distinct from the multiple point itself — of two such curves having the same multiple point. Let the order of the second curve be n; then through the multiple point pass $n-1$ branches. The further points of intersection are then $mn-(m-1)(n-1)=m+n-1$ in number. On the other hand, if the first curve is given, the number of conditions which the second can fulfil is $2n$. This may be seen readily by choosing the multiple point as a vertex of the reference triangle, whereupon the equation of the $J^{(n)}$ reduces to $2n+1$ terms.

The most interesting case occurs when the number of free intersections, $m+n-1$, is greater by one than the number of conditions that can be imposed on the second curve; that is, when $n=m-2$. In this case we have an involution, I_1^{2m-3}, set up on the first curve. Interpreting two well-known theorems relating to involutions, we have the following geometrical properties, which will be found useful in what follows:

1) *There exist on a $J^{(m)}$ exactly $2m-3$ points having the property that a $J^{(m-2)}$, with the same multiple point as the $J^{(m)}$, can be drawn to have $(2m-3)$-point contact in each with the $J^{(m)}$.*

2) *These $2m-3$ points all lie on a $J^{(m-2)}$ with the same multiple point.*

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2. We now take the curve in parametric form. Using homogeneous coördinates, let the multiple point be the vertex A_0, with coördinates

$$x_0 = 1, \qquad x_1 = 0, \qquad x_2 = 0,$$

of the triangle of reference. The quantity $x_1/x_2 = \lambda$ may then serve as the parameter.

A slight change in notation, too, will be found convenient. Since we shall be considering as fundamental the form giving the values of the parameter at the coincidence-points of the involution, it will be better if we take the order of this form to be $2m - 1$. The curve is then a $J^{(m+1)}$; its multiple point is of order m; and the curve whose intersections with it give the fundamental form is a $J^{(m-1)}$.

Interpretation of apolar forms.

3. Another property of an involution, important for our purpose, is this: Any $(2m - 1)$-ic whose roots form a set in the involution is apolar to the $(2m-1)$-ic giving the coincidence-points. Interpreted geometrically, this says that *the form which, equated to zero, gives the intersections of any $J^{(m-1)}$ with the original $J^{(m+1)}$ is apolar to the form whose vanishing gives the coincidence-points.* For brevity, let us call the $J^{(m-1)}$ which cuts out the coincidence-points the *covariant* $J^{(m-1)}$.

Now we know that there exist forms of lower order than the $(2m - 1)$-th, apolar to a $(2m - 1)$-ic; in fact, the apolarity of a form of order k involves the identical vanishing of a $(2m - 1 - k)$-ic, and therefore $2m - k$ conditions upon the k independent coefficients of the k-ic. These $2m - k$ conditions can then always be satisfied when $k \geqq 2m - k$, or $k \geqq m$. When $k = m$ there is a unique form, the canonizant of the $(2m - 1)$-ic, which has the property specified.

Geometrically, it is easy to see what corresponds to these apolar forms of lower than the $(2m - 1)$-th order. For a $J^{(m-1)}$ may degenerate in various ways; thus, it may become a $J^{(m-2)}$ and a line, which would have to pass through the multiple point, but would be otherwise arbitrary. That is, the $2m - 2$ points in which the $J^{(m+1)}$ is cut by the $J^{(m-2)}$, with any further point whatever, would be given by a form apolar to the $(2m - 1)$-ic; but this is equivalent to saying that the form which gives the $2m - 2$ points is apolar to the $(2m - 1)$-ic.

But further, a $J^{(m-3)}$ and any two lines through A_0 constitute a $J^{(m-1)}$; the $2m - 3$ intersections of the $J^{(m+1)}$ and the $J^{(m-3)}$ are accordingly given by a form of order $2m - 3$, apolar to the $(2m - 1)$-ic.

Making the $J^{(m-1)}$ degenerate thus step by step, we shall get apolar forms of successively lower order. The last step occurs when the $J^{(m-1)}$ has become a $J^{(1)}$ and $m - 2$ lines through A_0. Now a $J^{(1)}$ is a line having A_0 as a point of

zero order; that is, not passing through it at all. Hence the $m + 1$ points of intersection of any line with the $J^{(m+1)}$ are the vanishing points of an apolar form of order $m + 1$.

Lastly, *the unique apolar m-ic is given by the m values of the parameter at* A_0 *itself.* For, drawing an arbitrary line through this point, we have seen that the $m + 1$ intersections determine an apolar $(m + 1)$-ic; but of these intersections, m are now at A_0, and the remaining one is arbitrary. We have accordingly the complete interpretation of the apolar forms.

Introduction of the syzygies.

4. So far we have made no further choice of a reference triangle than to specify the position of the multiple point. It now becomes necessary to choose the opposite line; beyond this the triangle shall remain, for the present, arbitrary. For the side $x_0 = 0$ of the reference triangle we should naturally choose a line which cuts out a set of points forming a simple covariant of the fundamental $(2m - 1)$-ic; for example, the following: * Every odd form $C_{1, 2m-1}$ has a covariant $C_{2, 2}$; let the line joining the points which are roots of this covariant be chosen. It cuts out $m - 1$ other points which form a $C_{3(m-1), m-1}$, the canonizant of the $C_{3, 2m-3}$ obtained by operating with the $C_{2, 2}$ upon the fundamental $C_{1, 2m-1}$.

The equation of the $J^{(m+1)}$ can now be written down. It must be, to begin with, of the form

$$x_0 f_m(x_1, x_2) = f_{m+1}(x_1, x_2),$$

the f's being homogeneous functions of orders denoted by subscripts. But both f's are known; f_m is the product of the m tangents to the curve at A_0, or in other words, considered as an expression in x_1/x_2 or λ, it is the canonizant, $C_{m, m}$, of the $C_{1, 2m-1}$. Similarly f_{m+1} is the product $C_{2, 2} C_{3(m-1), m-1}$ of the two covariants cut out by the side $x_0 = 0$. We have accordingly the following equation of the $J^{(m+1)}$: $x_0 C_{m, m} = C_{2, 2} C_{3(m-1), m-1}$.

The covariant curve $J^{(m-1)}$ will have an equation of the form

$$x_0 f_{m-2}(x_1, x_2) = f_{m-1}(x_1, x_2),$$

where f_{m-2} is the form (covariant, naturally, to the $C_{1, 2m-1}$) which gives the points of the $J^{(m+1)}$ cut out by the tangents to the $J^{(m-1)}$ at A_0; f_{m-1}, those cut out by the lines joining A_0 to the points of intersection of the $J^{(m-1)}$ with $x_0 = 0$. So far, both of these are unknown; but consider the result of eliminating x_0 between the two equations. We shall get

$$f_{m-1} C_{m, m} - f_{m-2} C_{2, 2} C_{3(m-1), m-1} = 0.$$

On the other hand, however, we know that the intersections are given by the fundamental $C_{1, 2m-1}$. Hence *the* $C_{1, 2m-1}$, —*multiplied, in order to secure*

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* This choice was proposed by Professor Morley in suggesting the subject.

homogeneity, by a suitable invariant, whose value we shall presently derive,— must be identically expressible in the form

$$f_{m-2} C_{2,2} C_{3(m-1),m-1} - f_{m-1} C_{m,m};$$

in other words, there exists a syzygy

$$\lambda C_{1,2m-1} I + \mu f_{m-2} C_{2,2} C_{3(m-1),m-1} + \nu f_{m-1} C_{m,m} = 0.$$

5. To derive the form of the invariant which occurs in the first term, observe that if it vanishes, the canonizant $C_{m,m}$ must have at least one root in common with the product $C_{2,2} C_{3(m-1),m-1}$; but this means geometrically that the line chosen for the side $x_0 = 0$ passes through A_0, in which case the canonizant is a factor of $C_{2,2} C_{3(m-1),m-1}$, and has, not one, but all m roots in common with that product. Thus the theorem is suggested that *if the $C_{m,m}$ has a single root in common with the product $C_{2,2} C_{3(m-1),m-1}$, it must have all m, that is, must become a factor of that covariant product;* and this theorem might be readily verified algebraically.

A generalization is easily obtained. Let ϕ_{m+1} be any $(m+1)$-ic apolar to the $C_{1,2m-1}$; then * its $m+1$ roots lie on a line. If now ϕ_{m+1} and $C_{m,m}$ have a root in common, this line must go through A_0, and the $C_{m,m}$ becomes a factor of ϕ_{m+1}. This theorem, too, is easy to verify algebraically; inasmuch, however, as it is geometrically evident, further proof seems unnecessary.

6. From the preceding paragraph it will appear probable that *the proper invariant multiplier (of the $C_{1,2m-1}$ in the syzygy) is merely the eliminant of the $C_{2,2}$ and the $C_{m,m}$.* At least this eliminant, I_{4m}, is a factor in the invariant we seek. It might conceivably happen that it would have to be raised to some higher power than the first, but clearly no other invariant can enter as a factor. Moreover, it may be added that in all the cases examined, the first power of the eliminant has sufficed.

Assuming, then, this I_{4m} as the proper multiplier, we see that the degree of the syzygy is $4m + 1$; but then we have immediately the degrees of the two unknown f's. Finally, the syzygy takes the form

$$\lambda C_{1,2m-1} I_{4m} + \mu C_{2,2} C_{3(m-1),m-1} C_{m+2,m-2} + \nu C_{m,m} C_{3m+1,m-1} = 0.$$

7. Take next the line cutting out the zeros of a quadratic covariant, $C_{2k,2}$, as the side $x_0 = 0$ of our reference triangle; and inquire what form the corresponding syzygy would take. We should then have

$$J^{(m+1)}: \quad x_0 C_{m,m} = C_{2k,2} C_{(2k+1)(m-1),m-1},$$

$$J^{(m-1)}: x_0 C_{m+2,m-2} = C_{2km+m+1,m-1}.$$

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*See § 3.

To justify these equations, note that the remaining covariant cut out by the line $x_0 = 0$ must be the canonizant of the $C_{2k+1,\, 2m-3}$ obtained by operating with the $C_{2k,\, 2}$ upon the fundamental form. As for the $J^{(m-1)}$, its tangents at A_0 must be given by the same covariant, no matter what line is chosen as $x_0 = 0$; then considerations of homogeneity enable us to fill in the right-hand side of the equation. Hence, as before, the $C_{1,\, 2m-1}$, multiplied by a suitable invariant, must be identically expressible in the form

$$C_{m,\, m}\, C_{2km+m+1,\, m-1} - C_{2k,\, 2}\, C_{(2k+1)(m-1),\, m-1}\, C_{m+2,\, m-2}\,.$$

Each term being of degree $2km + 2m + 1$, it follows that the invariant which multiplies the $C_{1,\, 2m-1}$ must be of degree $2km + 2m$; but this is just the degree of the eliminant of the $C_{m,\, m}$ and the $C_{2k,\, 2}$. This eliminant is therefore again the required multiplier (it must, by an argument similar to that previously used, be at least a factor, and its degree is right), and the syzygy is then

$$\lambda\, C_{1,\, 2m-1}\, I_{m(2k+2)} + \mu\, C_{2k,\, 2}\, C_{(2k+1)(m-1),\, m-1}\, C_{m+2,\, m-2} + \nu\, C_{m,\, m}\, C_{2km+m+1,\, m-1} = 0\,.$$

An eliminant which is a perfect m-th power.

8. Lastly, let a line be chosen for $x_0 = 0$ cutting out the roots of an apolar covariant, $C_{n,\, m+1}$, which does not break up into the product of covariants of lower order. In this case we have

$$J^{(m+1)}:\qquad x_0\, C_{m,\, m} = C_{n,\, m+1}\,,$$

$$J^{(m-1)}:\quad x_0\, C_{m+2,\, m-2} = C_{n+2,\, m-1}\,.$$

Any term in the cross-product being of degree $m + n + 2$, the invariant multiplying $C_{1,\, 2m-1}$ should be of degree $m + n + 1$; but the eliminant of the $C_{m,\, m}$ and the $C_{n,\, m+1}$, which we should expect, *a priori*, to be the proper multiplier, is of degree $m(m + n + 1)$. This strongly suggests the following theorem: *Given a form of odd order $2m - 1$, the eliminant of any apolar form of order $m + 1$ and the canonizant (unique apolar form of order m) is a perfect m-th power.* By this is meant, of course, that its mth root is rational in the coefficients of each of the forms entering into its composition, and therefore indirectly (the apolar $(m + 1)$-ic being supposed a rational covariant of the $(2m - 1)$-ic) in the coefficients of the $(2m - 1)$-ic itself.

9. *Proof:* Let the given $(2m - 1)$-ic be

$$f \equiv \sum_{i=1}^{m} (a_i x)^{2m-1}, \qquad a_i x \equiv a_{i1} x_1 + a_{i2} x_2\,,$$

the a's being non-symbolic coefficients. Its canonizant, P, is $A(a_1 x)(a_2 x)\cdots(a_m x)$, A being independent of x. It will prove convenient to take

$$A \equiv \Pi\, (a_i a_j)^2 \quad (\, i = 1, 2, 3, \cdots, m-1\,;\, j = 2, 3, 4, \cdots, m\,;\, i < j\,)$$

Next let the $(m+1)$-ic apolar to f be

$$Q \equiv (Qx)^{m+1} = (Q_1 x_1 + Q_2 x_2)^{m+1},$$

in which the Q's are symbolic. Now the identity expressing that Q and f are apolar is

$$\sum_{i=1}^{m} (a_i Q)^{m+1} (a_i x)^{m-2} \equiv 0.$$

Operate on this identity successively with all possible products of $m-2$ factors chosen from the first $m-1$ of P; that is, with

$$\frac{(a_1 x)(a_2 x) \cdots (a_{m-1} x)}{a_i x} \qquad (i = 1, 2, 3, \cdots, m-1).$$

This gives

$$(a_i Q)^{m+1}(a_i a_1)(a_i a_2) \cdots (a_i a_{i-1})(a_i a_{i+1}) \cdots (a_i a_{m-1})$$
$$+ (a_m Q)^{m+1}(a_m a_1)(a_m a_2) \cdots (a_m a_{i-1})(a_m a_{i+1}) \cdots (a_m a_{m-1}) = 0$$
$$(i = 1, 2, 3, \cdots, m-1),$$

a set of $m-1$ equations.

The eliminant of P and Q is

$$E \equiv A^{m+1}(a_1 Q)^{m+1}(a_2 Q)^{m+1} \cdots (a_m Q)^{m+1}.$$

Substitute in this the values of $(a_1 Q)^{m+1}$, $(a_2 Q)^{m+1}$, $\cdots$, $(a_{m-1} Q)^{m+1}$, furnished us by the $m-1$ equations above in terms of $(a_m Q)^{m+1}$. Then

$$E = (-1)^{m-1} A^{m+1} [(a_m Q)^{m+1}]^m \frac{[(a_m a_1)(a_m a_2) \cdots (a_m a_{m-1})]^{m-2}}{\Pi (a_i a_j)^2}$$
$$(i = 1, 2, 3, \cdots, m-2; j = 2, 3, 4, \cdots, m-1; i < j).$$

If we call the denominator A', we see that A' is what A becomes when we drop all factors containing a_m.

Multiplying and dividing the value of E by

$$(a_m a_1)^2 (a_m a_2)^2 \cdots (a_m a_{m-1})^2,$$

we have

$$E = (-1)^{m-1} A^{m+1} [(a_m Q)^{m+1}]^m \frac{[(a_m a_1)(a_m a_2) \cdots (a_m a_{m-1})]^m}{A},$$

$$-E = (-1)^m A^m [(a_m Q)^{m+1}]^m [(a_m a_1)(a_m a_2) \cdots (a_m a_{m-1})]^m,$$

$$\sqrt[m]{-E} = -A (a_m Q)^{m+1} (a_m a_1)(a_m a_2) \cdots (a_m a_{m-1}).$$

This is of course only one of m possible expressions for $\sqrt[m]{-E}$; for in operating upon the identity $\sum_{i=1}^{m}(a_i Q)^{m+1}(a_i x)^{m-2} \equiv 0$ we have arbitrarily selected $(a_m x)$ as the factor to be excluded from the operating product. An interchange of the factors of P permutes these m expressions among themselves; hence

the value of $\sqrt[m]{-E}$ is unaltered by any such interchange; in other words, $\sqrt[m]{-E}$ *is rational in the coefficients of* P. As it is obviously rational also in those of Q, the theorem is proved.

<div align="center">Additional covariant curves.</div>

10. This completes the general discussion so far as the fundamental form $C_{1,\,2m-1}$ — or, what comes to the same thing, the covariant $J^{(m-1)}$ — is concerned. There remain, however, the apolar forms of order $m+1$, $m+2$, $\cdots$, $2m-1$, given by the intersections of the corresponding curves $J^{(1)}$, $J^{(2)}$, $\cdots$, $J^{(m-1)}$, with the fixed $J^{(m+1)}$. For instance, every form of odd order $2m-1$ has an apolar covariant $C_{3,\,2m-3}$, obtained by transvecting the $C_{2,\,2}$ upon it. Corresponding, therefore, to our first choice of the line $x_0 = 0$, we have a syzygy

$$\lambda C_{3,\,2m-3} I_{4m} + \mu C_{2,\,2} C_{3(m-1),\,m-1} C_{m+4,\,m-4} + \nu C_{m,\,m} C_{3m+3,\,m-3} = 0$$

where $m \gtreqless 4$; similarly, our second choice (line joining the zeros of a $C_{2k,\,2}$) will give

$$\lambda C_{3,\,2m-3} I_{m(2k+2)} + \mu C_{2k,\,2} C_{(2k+1)(m-1),\,m-1} C_{m+4,\,m-4} + \nu C_{m,\,m} C_{2km+m+3,\,m-3} = 0;$$

and the line cutting out the covariant $C_{n,\,m+1}$ gives

$$\lambda C_{3,\,2m-3} I_{m+n+1} + \mu C_{n,\,m+1} C_{m+4,\,m-4} + \nu C_{m,\,m} C_{n+4,\,m-3} = 0.$$

11. Again, as a consequence of the geometrical principle that, given two distinct lines $u = 0$, $v = 0$, it is possible to choose λ so that $u + \lambda v = 0$ shall pass through any specified point (in particular, through the multiple point A_0), we have between the canonizant and any two $(m+1)$-ics apolar to the $C_{1,\,2m-1}$ an identity of the form

$$\lambda C_{n,\,m+1} + \mu C_{p,\,m+1} + \nu C_{m,\,m} C_{q,\,1} = 0.$$

But λ must be the eliminant of the $C_{p,\,m+1}$ and the $C_{m,\,m}$, that is, an I_{p+m+1}; similarly μ must be an I_{n+m+1}; hence the degree of the identity, or syzygy, is $m+n+p+1$, and the linear covariant is a $C_{n+p+1,\,1}$; the syzygy runs as follows:

$$C_{n,\,m+1} I_{p+m+1} + C_{p,\,m+1} I_{n+m+1} + C_{m,\,m} C_{n+p+1,\,1} = 0.$$

12. The above examples serve merely to indicate possibilities in finding syzygies of this type; in dealing with any special case, of course, many more can be discovered. This is notably true of the quintic, for which the fundamental irreducible system is well known.

There is one other point deserving of mention. In dealing with the case of the quintic, where a list of syzygies has been tabulated, we come upon certain ones which correspond to covariants of higher order than the quintic itself. Such syzygies, interpreted geometrically in the manner explained above, will still give

JONQUIÈRES curves whose intersections with the first curve are the roots of those covariants; but, except when the order of the covariant is precisely 6, the number of conditions imposed on the curve by making it cut out the roots of the covariant is not sufficient to determine the curve; in other words, the curve furnished us by the syzygy is not the only one of its type which would cut out the given covariant. Hence in such cases there is less interest attaching to the syzygy. On the other hand, the syzygy corresponding to any sextic covariant (in the general case, that corresponding to any covariant of order $2m$) is quite as important as that corresponding to any covariant of lower order.

We proceed now to the discussion of special cases.

The quintic.

13. The first special case presenting interest is that of the quintic. Here $m = 3$; the fixed curve is therefore a quartic, $J^{(4)}$, and the curve which cuts out the quintic is a conic, $J^{(2)}$.

For representing the concomitants of the quintic we shall make use of the notation of STROH,* in his paper on the syzygies of the quintic. For convenience of reference we reproduce the list, with their definitions, in the following table:

Degree.

1. f.

2. $(ff)^2 \equiv H$, $(ff)^4 \equiv i$.

3. $(fH) \equiv T$, $(fi) \equiv q$, $-(fi)^2 \equiv j$.

4. $(iH) = (jf) \equiv m$, $(fj)^2 \equiv h$, $(ii)^2 \equiv A$.

5. $(jH) = (fh) \equiv r$, $(ji) \equiv \epsilon$, $-(ji)^2 \equiv \alpha$.

6. $(hi) = \frac{1}{2}(fa) \equiv \eta$, $(jj)^2 = -(hi)^2 \equiv \tau$.

7. $(f\tau) = (jh) \equiv n$, $(i\alpha) \equiv \beta$.

Degree.

8. $(i\tau) = (\alpha j) \equiv \theta$, $(i\tau)^2 \equiv B$.

9. $(j\tau) \equiv Q$.

11. $(\tau\alpha) \equiv \gamma$.

12. $(\alpha\beta) \equiv C$.

13. $(i\gamma) \equiv \delta$.

18. $(\beta\gamma) = (\delta\alpha) \equiv R$.

As forms sometimes used instead of the foregoing we have

$$n' \equiv (H\alpha) = -n - i\epsilon, \quad C' \equiv (\tau\tau)^2 = \tfrac{1}{3}(C + 2AB), \quad k \equiv (iH)^2 = -h - \tfrac{3}{10}i^2.$$

14. If we choose as the line $x_0 = 0$ that cutting out the covariant i, the remaining pair of points on it are the roots of τ. Then we have for the equation of the $J^{(4)}$ $x_0 j = i\tau$. Now the eliminant of j and i may be readily proved to be the invariant C' (which is also the discriminant of j) as follows. First, the eliminant of a $C_{3,3}$ and a $C_{2,2}$ is of degree 12.† Taking f in the form

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*Mathematische Annalen, vol. 34 (1888), p. 354. This notation coincides with that of CLEBSCH in his Binäre Formen except in one particular; he defines the fourth linear covariant as $(\theta\alpha)$.

† See SALMON, Higher Algebra, Lesson VIII.

$(a, 0, 0, 0, e, f)(x, y)^5$, for which C' is known to vanish, we find *

$$j \equiv ae^2 xy^2; \qquad i \equiv aex^2 + afxy;$$

so that i and j have in common the factor x.

We must therefore look for a syzygy

$$\lambda f C' + \mu i\tau C_{5,1} + vj C_{10,2} = 0.$$

There being but a single $C_{5,1}$, we have everything but the numerical multipliers and the $C_{10,2}$. In fact we find in STROH's list the following two:

$$(f\alpha\beta): \qquad\qquad f C + 2\eta\beta - j\alpha^2 + Ah\alpha + \tfrac{3}{2} i\tau\alpha = 0,$$

$$(hi\beta): \ \eta\beta - fAB + \tfrac{1}{2}Ah\alpha - 2Aj\tau - \tfrac{3}{2}i\tau\alpha - \tfrac{1}{2}ijB = 0.$$

Subtracting twice the second from the first, and noting that $3C' = C + 2AB$, we have

$$3fC' + \tfrac{9}{2} i\tau\alpha + j(4A\tau - \alpha^2 + iB) = 0.$$

15. For a second choice of $x_0 = 0$ let us take the line cutting out the quadratic covariant θ of degree 8. The invariant that multiplies f (eliminant of j and θ), being of degree 30, a number not divisible by 4, must contain the skew invariant R as a factor; the remaining factor, an I_{12}, is readily proved to be C' once more. In fact, with f chosen as in § 14, $\theta \equiv 2a^3 e^5 xy + a^3 e^4 fy^2$, and has in common with j the factor y. We have therefore to look for a syzygy

$$\lambda f C' R + \mu\theta\alpha C_{18,2} + vj C_{28,2} = 0.$$

Take the following, given by STROH:

$$(\tau\beta\gamma) \times i: \ \ R i\tau - \gamma\delta i + \tfrac{1}{2}i\alpha\beta C' - i\alpha\gamma B \qquad\qquad\qquad = 0,$$

$$\{\tau\alpha\iota\gamma\}_2 \times -i: \quad\ \ - \gamma\delta i - \tfrac{1}{2}i\alpha\beta C' \qquad -\frac{\theta i}{2}(B^2 - AC') \qquad = 0,$$

$$(j\alpha\beta) \times 2\gamma: \quad\ 2\gamma\delta i - 2\alpha^3\gamma + i\alpha\gamma B - 4\beta\gamma\theta + 2\gamma Cj \qquad\quad = 0,$$

$$(j\delta\alpha) \times 2\alpha: \qquad\qquad 2\alpha^3\gamma \qquad\quad + 2\alpha\delta\theta + \alpha^2 B\theta + 2\alpha Rj = 0.$$

Adding, we have

$$R i\tau + \theta\left[\alpha^2 B + 2\alpha\delta - 4\beta\gamma - \frac{i}{2}(B^2 - AC')\right] + 2j(\alpha R + \gamma C) = 0.$$

In this the coefficients of both θ and j can be put into simpler form. In fact, adding in

$$(\alpha\iota\gamma) \times -\theta: \ \theta\left[-\alpha\delta + \beta\gamma + \tfrac{1}{2}iB^2 - \tfrac{1}{6}iAC - \tfrac{1}{3}iA^2 B\right] = 0,$$

$$(\beta\tau\alpha) \times -\theta: \ \theta\left[-\alpha^2 B - \alpha\delta - \beta\gamma - C\tau\right] \qquad\qquad = 0,$$

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* These values are taken from ELLIOTT, *Algebra of Quantics*, p. 309.

we have

$$Ri\tau - \theta(4\beta\gamma + C\tau) + 2j(\alpha R + \gamma C) = 0.$$

Finally, making use of

$$(\alpha\beta\gamma) \times - 2j: \; -2j\left(\alpha R + \gamma C + \frac{\beta}{2}[AC' - B^2]\right) = 0,$$

we find

$$Ri\tau - \theta(4\beta\gamma + C\tau) + j\beta(B^2 - AC') = 0.$$

This, while not the syzygy desired, will be found to be important in another connection. Moreover, on combining it with that previously obtained for f (end of § 14), we obtain at once the one which we are seeking; it is

$$3fC'R + \tfrac{9}{2}\theta\alpha(4\beta\gamma + C\tau) + j\left(-\frac{9\alpha\beta}{2}[B^2 - AC'] + R[4A\tau - \alpha^2 + iB]\right) = 0.$$

16. Since there are two independent $C_{4,4}$'s, h and i^2, but only one $C_{5,1}$, it must be possible to find a linear combination of these two which will be apolar to f; this will then certainly be the apolar quartic of lowest degree in the coefficients. The eliminant of it and j will be of degree 24, and therefore * the cube of an I_8. We are then to look for a syzygy

$$\lambda fI_8 + \mu\alpha(h - \rho i^2) + \nu jC_{6,2} = 0;$$

and we find in STROH's list the very syzygy sought,

$$[fhii]_4: \; fB + \alpha(\tfrac{1}{2}i^2 - h) + j(2\tau + \tfrac{1}{3}iA) = 0.$$

Corresponding to the three syzygies thus far obtained we shall have three forms of the equation of the covariant $J^{(2)}$. The results are conveniently summed up in the following table:

$J^{(4)}$	Syzygy.	$J^{(2)}$
$x_0j = i\tau$	$3fC' + \tfrac{9}{2}i\tau\alpha + j(4A\tau - \alpha^2 + iB) = 0$	$9x_0\alpha + 2(4A\tau - \alpha^2 + iB) = 0$
$x_0j = \theta(4\beta\gamma + C\tau)$	$3fC'R + \tfrac{9}{2}\theta\alpha(4\beta\gamma + C\tau)$ $+ j\left(-\dfrac{9\alpha\beta}{2}[B^2 - AC'] + R[4A\tau - \alpha^2 + iB]\right) = 0$	$9x_0\alpha - 9\alpha\beta(B^2 - AC')$ $+ 2R(4A\tau - \alpha^2 + iB) = 0$
$x_0j = h - \tfrac{1}{2}i^2$	$fB + \alpha(\tfrac{1}{2}i^2 - h) + j(2\tau + \tfrac{1}{3}iA) = 0$	$3x_0\alpha = 6\tau + iA$

Apolar quartics.

17. We look next for syzygies connecting the canonizant j and any two quartics apolar to f. Three such quartics are already found, namely,

$$i\tau, \qquad \theta(4\beta\gamma + C\tau), \qquad h - \tfrac{1}{2}i^2;$$

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* See § 8.

and in the process of deriving the second syzygy for f we incidentally developed the syzygy connecting the first two; * it was in fact

$$R i\tau - \theta (4\beta\gamma + C\tau) + j\beta (B^2 - A C') = 0.$$

In STROH's list we find

$$(h\alpha\beta): \quad 3 C'(h - \tfrac{1}{2}i^2) + \tfrac{9}{2}i\tau B + j(4\delta + B\alpha) = 0,$$

which connects the first and third. From these two, by elimination of $i\tau$, it would be easy to arrive at the syzygy between the two remaining quartics.

With each of these quartics is associated the invariant whose vanishing expresses that it contains the canonizant as a factor. Thus with the $C_{4,4}$ is paired the I_8, B; with the $C_{8,4}$, $i\tau$, the I_{12}, C'; and with the $C_{26,4}$, $\theta(4\beta\gamma + C\tau)$, the I_{30}, $C'R$. In every case we observe that the degree of the invariant is greater by 4 than that of the quartic. We naturally inquire whether apolar quartics can be found which pair with other known invariants; for example, does there exist a $C_{14,4}$ which will contain j as a factor on the sole condition $R = 0$? We shall prove the existence of such a $C_{14,4}$ by deriving a syzygy

$$\lambda C_{14,4} B + \mu R (h - \tfrac{1}{2}i^2) + \nu j C_{19,1} = 0.$$

From STROH's paper

$$(h\beta\gamma): \quad hR + \tfrac{3}{2}\alpha\tau\gamma - \tfrac{1}{2}i\alpha\beta B - \tfrac{1}{2}Ai\alpha\gamma$$
$$+ j\left(\frac{C\beta}{6} - \frac{\gamma}{6}[2A^2 - 3B]\right) = 0,$$

$$(\beta i\gamma) \times \frac{i}{2}: \quad -\tfrac{1}{2}i^2 R + \tfrac{1}{2}i\beta\delta \qquad + \tfrac{1}{4}Ai\alpha\gamma \qquad = 0,$$

$$\{i\tau\alpha\beta\}_2 \times -\frac{i}{2}: \quad -\tfrac{1}{2}i\beta\delta - \tfrac{1}{2}i\alpha\beta B + \tfrac{1}{4}Ai\alpha\gamma - \tfrac{1}{2}i\theta C \qquad = 0,$$

$$[ji\tau\tau]_3 \times -\frac{3\alpha}{2}: \quad -\tfrac{3}{2}\alpha\tau\gamma \qquad + \tfrac{3}{2}\alpha QB - \tfrac{3}{2}\alpha\epsilon C' \qquad = 0,$$

$$(j\tau\alpha) \times -\frac{3B}{2}: \quad -\tfrac{3}{2}\tau\theta B - \tfrac{3}{2}\alpha QB \qquad - \tfrac{3}{2}j\gamma B = 0,$$

$$(ji\alpha) \times \frac{3C'}{2}: \quad \tfrac{3}{2}i\theta C' \qquad + \tfrac{3}{2}\alpha\epsilon C' + \tfrac{3}{2}j\beta C' = 0.$$

Summation gives:

$$(h - \tfrac{1}{2}i^2)R + B(i\theta A - \tfrac{3}{2}\tau\theta - i\alpha\beta) + j\left(\beta\left[\frac{C + 9C'}{6}\right] - \frac{\gamma}{6}[2A^2 + 6B]\right) = 0.$$

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*See § 15.

Below are collected the three syzygies of this kind that have been derived. They contain implicitly the complete relations between the four apolar quartics.

(1) $$3\,C'(h - \tfrac{1}{2}i^2) + \tfrac{9}{2}i\tau B + j\,(4\delta + B\alpha) = 0\,,$$

(2) $$Ri\tau - \theta\,(4\beta\gamma + C\tau) + j\beta\,(B^2 - A\,C') = 0\,,$$

(3) $$(h - \tfrac{1}{2}i^2)R + B(i\theta A - \tfrac{3}{2}\tau\theta - i\alpha\beta) + j\left(\beta\left[\frac{C + 9\,C'}{6}\right] - \frac{\gamma}{6}\,[2A^2 + 6B]\right) = 0\,.$$

We wish to point out the existence of a syzygy of another kind which, as theory suggests, must exist between the four quartics alone. In fact, between the equations of any four lines in a plane there must exist an identity; hence there is a four-term syzygy in which each term is one of the quartics multiplied by the invariant whose vanishing expresses that the remaining three are in an involution.

18. Below are given all the remaining syzygies that have been derived for the quintic. Most of them have been verified, either directly or by combination, from STROH's paper.

H: $$6\,HC' - 9i\tau^2 - j\,(4jB + 6\tau\alpha) = 0\,,$$

H: $$HB + (h - \tfrac{1}{2}i^2)\tau - \tfrac{2}{3}j\,(2i\alpha + jA) = 0\,,$$

ϵ^2: $$2\epsilon^2 + i^2\tau + j\,(2i\alpha + jA) = 0\,,$$

$\epsilon\theta$: $$2\epsilon\theta - i\tau\alpha + j\,(iB + \tau A) = 0\,,$$

ϵQ: $$2\epsilon Q + i\tau^2 + j\,(jB + \tau\alpha) = 0\,,$$

$h\alpha$: $$3\,C'h\alpha + i\tau\,(\tfrac{3}{2}\delta + \tfrac{15}{4}\alpha B) + j(\tfrac{3}{4}i\,[A\,C' - B^2] + \alpha\,[4\delta + B\alpha\,]) = 0\,,$$

m: $$3m\,C' - \tfrac{9}{2}i\tau\theta + j\,(i\gamma - 3\tau\beta + 2\epsilon B) = 0\,,$$

r: $$r\,C' - \tfrac{3}{2}i\tau Q + j\,(\tfrac{1}{3}j\gamma + \tau\theta) = 0\,,$$

T: $$TC' - \tfrac{3}{2}i\tau n + j\,(\tfrac{2}{3}mB - 2\tau\eta + \tfrac{1}{3}f\gamma) = 0\,.$$

Eliminants with multiple factors.

19. An examination of the syzygies, in the cases where some one or other of the invariants is zero, leads readily to many interesting properties of the covariants. Thus when $B = 0$ we find that τ is a factor of both ϵ and H (and therefore that these two latter have *two* roots in common); when $C' = 0$, that j, τ and H all have the same double root, which is also a simple root of ϵ, θ and γ; that the remaining root of j is also a root of i and ϵ; and so on. On calculating the eliminants of certain pairs of covariants, and considering them in connection with the results just obtained, some curious facts are brought to light. For

instance, we find by calculation that the eliminant of τ and ϵ is $B^2 C'$; on examining the results above it appears that when $B = 0$ they have *two* roots in common, while when $C' = 0$ they have *one* only. Again, the eliminant of τ and H is $B^2 C'^2$; this accords perfectly with the circumstance that when either invariant vanishes they have *two* roots in common. Several other instances might be adduced which seem to point to the same general principle, namely, that if the eliminant of two covariants, P and Q, contains as a factor the rth power of a certain invariant I, then when I vanishes, P and Q have r roots in common; it appears, however, that the principle is not universally valid. It might, perhaps, if properly limited (say by requiring that P and Q have no multiple roots), be made to give exact information.

Forms of order higher than the fifth.

20. In conclusion, a brief summary of the results arrived at in the case of the septimic will be given. First, a mode of procedure entirely different from that adopted in the preceding case was necessary. While a complete irreducible system of concomitants has been worked out,* their number is very great, and moreover no list of syzygies has ever been tabulated. It seemed best, therefore, to attack the problem by choosing a special septimic, with several coefficients zero, and calculating out the syzygies for this form; then finally, when all the covariants occurring therein are known, verifying the most important of the syzygies for a general form.

The syzygies to be looked for are as follows:

$$(1) \qquad C_{1,7} I_{16} + C_{2,2} C_{9,3} C_{6,2} + C_{4,4} C_{13,3} = 0,$$

$$(2) \qquad C_{1,7} I_8 + C_{3,5} C_{6,2} + C_{4,4} C_{5,3} = 0,$$

$$(3) \qquad C_{3,5} I_{16} + C_{2,2} C_{9,3} I_8 + C_{4,4} C_{15,1} = 0,$$

$$(4) \qquad C_{2,6} I_8 + C_{3,5} C_{7,1} + C_{4,4} C_{6,2} = 0.$$

The first is the syzygy obtained for the fundamental form itself when the line cutting out the covariant $C_{2,2}$ is taken for the side $x_0 = 0$.† The unknown covariants in it are $C_{6,2}$ and $C_{13,3}$. So (2) is that obtained by having $x_0 = 0$ cut out the covariant $C_{3,5}$;‡ (3) gives the connection between the two apolar quintics and the canonizant.§ All four of these syzygies have been worked out for the form

$$f \equiv a x_1^7 + 21 c x_1^5 x_2^2 + 7 g x_1 x_2^6 + h x_2^7,$$

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* By VON GALL, Mathematische Annalen, vol. 31 (1887), p. 318.

† See §§ 4–6.

‡ See § 8.

§ See § 11.

and the second of them has been verified for the general form — this particular syzygy being chosen as that of lowest degree containing f itself in its first term. The most interesting part of the result is that the I_8, occurring not only in that syzygy, but in (3) and (4) as well, is that invariant whose vanishing is the condition for self-apolarity of the canonizant, so that we have the following theorem : *If a septimic has a self-apolar canonizant, the latter is a factor of the covariant $C_{3,5}$.*

Even in the case of the septimic the method involves laborious calculations ; when we advance to the ninth and higher orders, for which no irreducible system of concomitants has been tabulated, its application becomes impracticable.